

Anisotropic Born-Infeld Cosmologies

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Abstract

Anisotropic cosmological spacetimes are constructed from spherically symmetric solutions to Einstein's equations coupled to nonlinear electrodynamics and a positive cosmological constant. This is accomplished by finding solutions in which the roles of r and t are interchanged for all $r > 0$ (i.e. r becomes timelike and t becomes spacelike). Constant time hypersurfaces have topology $R \times S^2$ and in all the spacetimes considered the radius of the two sphere vanishes as t goes to zero. The scale factor of the other dimension diverges as t goes to zero in some solutions and vanishes (or goes to a constant) in other solutions. At late times local observers would see the universe to be homogeneous and isotropic.

Introduction

Over the last few years Born-Infeld theory [1] has undergone a revival due to its appearance in string theory [2]. In this paper some exact cosmological solutions are found to the Einstein field equations coupled to nonlinear electrodynamics and a positive cosmological constant. These solutions are constructed from spherically symmetric solutions with $g_{tt} = 1/g_{rr} = -(1 - 2m(r)/r)$. If $m(r) > \frac{1}{2}r$ for $0 < r < \infty$ then r and t interchange roles and the solutions describe cosmological spacetimes with a singularity at $t = 0$ (instead of at $r = 0$). Constant time hypersurfaces have topology $R \times S^2$ and the radius of the two sphere goes to zero as t goes to zero. The scale factor of the other dimension diverges as t goes to zero in some solutions and vanishes (or goes to a constant) in other solutions. The Schwarzschild solution with a cosmological constant leads to a cosmological solution as does Born-Infeld theory. However, Maxwell's theory does not as it is not possible to satisfy $m(r) > \frac{1}{2}r$ for all r on $(0, \infty)$. Some other Born-infeld cosmologies can be found in [3, 4].

Born-Infeld Theory

In nonlinear electrodynamics the Maxwell Lagrangian

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(E^2 - B^2) \quad (1)$$

is replaced by

$$L = L(F^2, G^2) \quad (2)$$

where $F^2 = \frac{1}{2}F^{\mu\nu}F_{\mu\nu}$, $G^2 = \frac{1}{2}F^{\mu\nu}F_{\mu\nu}^*$, $F_{\mu\nu}^*$ is the dual of $F_{\mu\nu}$, and L is any function that reduces to (1) in the weak field limit. Born and Infeld took L to be given by

$$L = -\frac{1}{a^2} [\sqrt{1 + a^2 F^2} - 1] \quad (3)$$

For the solutions considered in this paper $\vec{B} = 0$ so that $G^2 = 0$. Thus, all G^2 dependences will be dropped.

The field equations are

$$\nabla_\mu P^{\mu\nu} = 0 \quad (4)$$

and

$$\nabla_\mu F^{*\mu\nu} = 0, \quad (5)$$

where

$$P^{\mu\nu} = \frac{\partial L}{\partial F_{\mu\nu}}. \quad (6)$$

The energy-momentum tensor is

$$T^{\mu\nu} = -2P^{\mu\alpha}F^\nu{}_\alpha + g^{\mu\nu}L \quad (7)$$

and the “Hamiltonian”, which is a function of $P^{\mu\nu}$, is

$$H = P^{\mu\nu} F_{\mu\nu} - L. \quad (8)$$

For the Born-Infeld Lagrangian

$$T^{\mu\nu} = \left[\frac{F^{\mu\alpha} F_{\alpha}^{\nu}}{\sqrt{1 + a^2 F^2}} - \frac{1}{a^2} g^{\mu\nu} (\sqrt{1 + a^2 F^2} - 1) \right] \quad (9)$$

and

$$H(P^2) = \frac{1}{a^2} [\sqrt{1 + a^2 P^2} - 1] \quad (10)$$

where $P^2 = -2P^{\alpha\beta}P_{\alpha\beta}$.

Cosmologies from Spherically Symmetric Solutions

Birkhoff’s theorem holds for nonlinear electrodynamic theories and the general spherically symmetric solution is [5, 6, 7, 8, 9, 10, 11]

$$ds^2 = - \left[1 - \frac{2m(r)}{r} \right] dt^2 + \left[1 - \frac{2m(r)}{r} \right]^{-1} dr^2 + r^2 d\Omega^2 \quad (11)$$

$$P = \frac{Q}{r^2} dt \wedge dr \quad (12)$$

and

$$\frac{dm(r)}{dr} = 4\pi r^2 H(P^2) + \frac{1}{2} r^2 \Lambda \quad (13)$$

where $P^2 = Q^2/r^4$ and Λ is the cosmological constant.

If $m(r) > \frac{1}{2}r$ for $0 < r < \infty$ then r is a timelike coordinate and t is a spacelike coordinate. Relabeling r and t and denoting the spacelike variable by x gives

$$ds^2 = - \left[\frac{2m(t)}{t} - 1 \right]^{-1} dt^2 + \left[\frac{2m(t)}{t} - 1 \right] dx^2 + t^2 d\Omega^2 \quad (14)$$

$$P = \frac{Q}{t^2} dx \wedge dt \quad (15)$$

and

$$\frac{dm(t)}{dt} = 4\pi t^2 H \left[\frac{Q^2}{t^4} \right] + \frac{1}{2} t^2 \Lambda. \quad (16)$$

Constant timelike surfaces have topology $R \times S^2$ and the two sphere has radius t . The Ricci scalar is given by

$$R = -2 \left[\frac{t\ddot{m} + 2\dot{m}}{t^2} \right] \quad (17)$$

and R generically diverges as t goes to zero.

Equation (16) can be written as

$$\frac{dm(t)}{dt} = 4\pi t^2 H \left[\frac{Q^2}{t^4} \right] + \frac{1}{2} t^2 \Lambda \quad (18)$$

Integrating gives

$$\frac{2m(t)}{t} - 1 = \frac{8\pi}{t} \int t^2 H \left[\frac{Q^2}{t^4} \right] dt + \frac{2m_0}{t} + \frac{1}{3} \Lambda t^2 - 1 \quad (19)$$

where m_0 is a constant. It is important to remember that the constraint

$$\frac{2m(t)}{t} - 1 > 0 \quad (20)$$

must be satisfied for all $t > 0$.

First consider the case $Q = 0$ and take $H(0) = 0$. The constraint becomes

$$\frac{2m_0}{t} + \frac{1}{3} \Lambda t^2 - 1 > 0. \quad (21)$$

This will be satisfied if $\Lambda > 0$ and $m_0 > \frac{1}{3} \Lambda^{-1/2}$. Even though R remains finite as t goes to zero the scalar $R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$ diverges, so that $t = 0$ is an initial singularity. Thus, Schwarzschild with a positive cosmological constant can be converted into a cosmological solution with metric

$$ds^2 = - \left[\frac{2m_0}{t} + \frac{1}{3} \Lambda t^2 - 1 \right]^{-1} dt^2 + \left[\frac{2m_0}{t} + \frac{1}{3} \Lambda t^2 - 1 \right] dx^2 + t^2 d\Omega^2. \quad (22)$$

As $t \rightarrow 0$ the two sphere collapses but the x direction blows up. For large t the metric is

$$ds^2 = -d\tau^2 + \exp \left[2\sqrt{\frac{\Lambda}{3}} \tau \right] \left[d\bar{x}^2 + d\Omega^2 \right], \quad (23)$$

where $\tau = \sqrt{3/\Lambda} \ln t$ and $\bar{x} = \sqrt{\Lambda/3} x$. Thus, at late times we have inflationary behaviour and the scale factor of the two sphere is the same as the scale factor for the x direction.

Next consider Maxwell's theory with $H(P^2) = 1/2 P^2 = Q^2/2t^4$. The constraint is

$$\frac{2m_0}{t} - \frac{4\pi Q^2}{t^2} + \frac{1}{3} \Lambda t^2 - 1 > 0 \quad (24)$$

which cannot be satisfied for all $t > 0$. The problem is that the Q^2 term diverges faster than the m_0 term and has the wrong sign. This can be modified in nonlinear electrodynamics by including a more divergent term with the correct sign or by eliminating the

divergence. If Maxwell's theory is modified so that $H(P^2) = \frac{1}{2}P^2 - \alpha^2 P^4$, the constraint becomes

$$\frac{2m_0}{t} - \frac{4\pi Q^2}{t^2} + \frac{8\pi\alpha^2 Q^4}{5t^6} + \frac{1}{3}\Lambda t^2 - 1 > 0. \quad (25)$$

This inequality is satisfied for a wide range of values of the parameters m_0, Q, Λ , and α . Here the additional term diverges more rapidly than the Maxwell term and has the correct sign. This spacetime behaves in a similar fashion to the case with $Q = 0$.

Finally consider the Born-Infeld Lagrangian. The constraint is

$$\frac{2m_0}{t} + \frac{1}{3}\Lambda t^2 - 1 + \frac{8\pi}{a^2 t} \int_0^t \left[\sqrt{a^2 Q^2 + x^4} - x^2 \right] dx > 0. \quad (26)$$

Since the integral is greater than zero for all $t > 0$ the inequality will certainly be satisfied if $m_0 > \frac{1}{3}\Lambda^{-1/2}$. In Born-Infeld theory the electric contribution remains finite and does not present a problem as t goes to zero. For $m_0 > 0$ this spacetime has similar properties to the case with $Q = 0$. It is possible to take $m_0 = 0$. For small t

$$\frac{2m(t)}{t} - 1 \simeq \frac{1}{3} \left[\Lambda - \frac{8\pi}{a^2} \right] t^2 - 1 + \frac{8\pi}{a} |Q|. \quad (27)$$

Thus, we require that $|Q| \geq a/8\pi$. Now let $f(t) = t(2m(t)/t - 1)$. The derivative of $f(t)$ is given by

$$f'(t) = \left[\Lambda - \frac{8\pi}{a^2} \right] t^2 - 1 + \frac{8\pi}{a^2} \sqrt{a^2 Q^2 + t^4}. \quad (28)$$

If $\Lambda \geq 8\pi/a^2$ and $|Q| \geq a/8\pi$ then $f'(t) > 0$ for $t > 0$ and $2m(t)/t - 1 > 0$ for $t > 0$.

Equation (19) determines the spacetime metric given $H(P^2)$. The reverse process is also possible. For a metric of the form

$$ds^2 = -\frac{dt^2}{a(t)^2} + a(t)^2 dx^2 + t^2 d\Omega^2 \quad (29)$$

the Hamiltonian is given by

$$H \left[\frac{Q^2}{t^4} \right] = \frac{1}{4\pi t^2} \left[\frac{d}{dt}(ta^2) - \Lambda t^2 + 1 \right]. \quad (30)$$

To be physically reasonable H must reduce to the Maxwell Hamiltonian in the weak field limit.

Conclusion

Exact cosmological solutions to the Einstein field equations coupled to nonlinear electrodynamics, including Born-Infeld theory, were constructed. These solution were produced by considering spherically symmetric solutions in which the roles of r and t are reversed.

These spacetimes have an initial singularity and constant time hypersurfaces have topology $R \times S^2$. The radius of the two sphere is t and the scale factor of the other dimension diverges in some cases as t goes to zero and vanishes (or goes to a constant) in other cases. At late times local observers would see the universe to be homogeneous and isotropic. Such solutions can be constructed from the Schwarzschild solution with a positive cosmological constant and from Born-Infeld theory. Maxwell theory does not lead to a cosmological solution because the roles of r and t cannot be reversed for all $r > 0$.

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